2016/17 MATH2230B/C Complex Variables with Applications
Suggested Solution of Selected Problems in HW 2
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P.71 3(b),4(b) will be graded

All the problems are from the textbook, Complex Variables and Application (9*th* edition).

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3. Form results obtain in Secs. 21 and 23, determine where $f'(z)$ exists and find its value when

   (a) $f(z) = 1/z$;

   (b) $f(z) = x^2 + iy^2$;

   (c) $f(z) = z\text{Im}(z)$.

   Solution. *Assume that $z = x + iy$, $x, y \in \mathbb{R}$ and denote*

$$f(z) = f(x + iy) = u(x, y) + iv(x, y).$$

   (a) Let $f(z) = 1/z$ and rewrite f to be

$$f(z) = f(x + iy) = \frac{1}{x + iy} = \frac{x}{x^2 + y^2} + i\left(\frac{-y}{x^2 + y^2}\right).$$

   For any $(x_0, y_0) \neq (0, 0)$, the function $u(x, y)$ and $v(x, y)$ are continuously differentiable at (x_0, y_0) and satisfy the Cauchy-Riemann equations:

$$\frac{\partial u}{\partial x}(x, y) = \frac{y^2 - x^2}{(x^2 + y^2)^2} = \frac{\partial v}{\partial y}(x, y),$$

$$\frac{\partial u}{\partial y}(x, y) = -\frac{2xy}{(x^2 + y^2)^2} = -\frac{\partial v}{\partial x}(x, y).$$

   Thus, $f'(z)$ exists at $z_0 = x_0 + iy_0$.
   For $(x_0, y_0) = (0, 0)$, the first-order partial derivatives of $u(x, y)$ and $v(x, y)$ do not exist at $(0, 0)$. Hence, $f'(0)$ does not exist.

   (b) For $f(z) = x^2 + iy^2$, we first calculate the first-order derivatives of $u(x, y)$ and $v(x, y)$:

$$\frac{\partial u}{\partial x}(x, y) = 2x, \quad \frac{\partial u}{\partial y}(x, y) = 0,$$

$$\frac{\partial v}{\partial x}(x, y) = 0, \quad \frac{\partial v}{\partial y}(x, y) = 2y.$$

   When $z = x + ix$, the Cauchy-Riemann equations is satisfied and hence $f'(z)$ exists and we have

$$f'(z) = f(x + ix) = 2x.$$

   For $z = x + iy$ with $x \neq y$, the Cauchy-Riemann equations is not satisfied and $f'(z)$ does not exist at those points.

(c) *For $f(z) = zIm(z)$, the first-order derivatives of $u(x, y)$ and $v(x, y)$ are*

$$\frac{\partial u}{\partial x}(x, y) = y, \quad \frac{\partial u}{\partial y}(x, y) = x,$$
$$\frac{\partial v}{\partial x}(x, y) = 0, \quad \frac{\partial v}{\partial y}(x, y) = 2y.$$

Hence, the Cauchy-Riemann equations can be satisfied only at $z = 0$ and $f'(0) = 0$. For $z \neq 0$, $f'(z)$ does not exist.

4. Use the theorem in Sec. 24 to show that each of these functions is differentiable in the indicated domain of definition, and also to find $f'(z)$:

   (a) $f(z) = 1/z^4$ $(z \neq 0)$;

   (b) $f(z) = e^{-\theta} \cos(\log r) + ie^{-\theta} \sin(\log r)$ $(r > 0, 0 < \theta < 2\pi)$.

Proof. For $z \neq 0$, we write $z = re^{i\theta}$ with $r > 0$ and $-\pi < \theta \leq \pi$ and $f(z) = u(r, \theta) + iv(r, \theta)$.

(a) By simple computation, for $f(z) = 1/z^4$, we have

$$u(r, \theta) = r^{-4} \cos(4\theta) \quad \text{and} \quad v(r, \theta) = -r^{-4} \sin(4\theta).$$

Also, the first-order derivatives of $u(r, \theta)$ and $v(r, \theta)$ with respect to r and θ exist and are continuous for (r, θ) with $r > 0$ and $\theta \in (-\pi, \pi]$. We compute them as follow

$$u_r = -4r^{-5} \cos(4\theta), \quad u_\theta = -4r^{-4} \sin(4\theta),$$
$$v_r = 4r^{-5} \sin(4\theta), \quad v_\theta = -4^{-4} \cos(4\theta).$$

Observe that for any (r, θ) with $r > 0$ and $-\pi < \theta \leq \pi$, the polar form of the Cauchy-Riemann equations are satisfied at (r, θ):

$$ru_r = -4r^{-4} \cos(4\theta) = v_\theta,$$
$$u_\theta = -4r^{-4} \sin(4\theta) = -rv_r.$$

Hence, $f'(z)$ exists and

$$f'(z) = e^{-i\theta}(-4r^{-5} \cos(4\theta) + i4r^{-5} \sin(4\theta)) = -4r^{-5}e^{-i5\theta} = \frac{-4}{z^5}.$$

(b) By simple calculation, we have the following results:

$$u(r, \theta) = e^{-\theta} \cos(\log r), \quad v(r, \theta) = e^{-\theta} \sin(\log r),$$
$$u_r = -e^{-\theta} \sin(\log r)/r, \quad u_\theta = -e^{-\theta} \cos(\log r),$$
$$v_r = e^{-\theta} \cos(\log r)/r, \quad v_\theta = -e^{-\theta} \sin(\log r).$$

Hence, the polar form of Cauchy-Riemann equations are satisfied at (r, θ) with $r > 0$ and $\theta \in (0, 2\pi)$

$$ru_r = v_\theta \quad \text{and} \quad u_\theta = -rv_r.$$

Also, we can calculate $f'(z)$ to obtain

$$f'(z) = e^{-i\theta}\left(- e^{-\theta}\frac{\sin \log r}{r} + ie^{-\theta}\frac{\cos \log r}{r}\right) = \frac{if(z)}{z}.$$

□

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5. Show that if the condition that $f(x)$ is real in the reflection principle (Sec. 29) is replaced by the condition that $f(x)$ is pure imaginary, then equation (1) in the statement of the principle is changed to

$$\overline{f(z)} = -f(\overline{z}).$$

Proof. As usual, we denote $f(z) = f(x+iy) = u(x,y) + iv(x,y)$.
Assume that $\overline{f(z)} = -f(\overline{z})$ is hold, then for $(x,0)$ on the segment of the real axis lies in D, we have

$$f(\overline{z}) = u(x,0) + iv(x,0) = -u(x,0) + iv(x,0) = -\overline{f(z)}.$$

It implies that $u(x,0) = 0$ and f is pure imaginary for each poitn x on the segment. Next, we assume that $f(x)$ is purely imaginary at each point x on the segment. Define $F(z) = \overline{f(\overline{z})}$ and similar to the theorem in Section 29, $F(z)$ is analytic in D and

$$F(z) = U(x,y) + iV(x,y) = u(x,-y) - iv(x,-y).$$

Since $f(x)$ is purely imaginary on the segment, then

$$F(x) = iV(x,0) = -iv(x,0) = -f(x).$$

By the uniqueness of the analytic function, we have

$$F(z) = -f(z) \quad \text{in } D.$$

This implies that

$$\overline{f(z)} = -f(\overline{z}),$$

□